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De novo genere serierum rationalium et valde convergentium, quibus ratio peripheriae ad diametrum exprimi potest

Leonhard Euler

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DE NOVO GENERE
SERIERVM RATIONALIVM
ET VALDE CONVERGENTIVM

QVIBVS
RATIO PERIPHERIAE AD DIAMETRVM
EXPRIMI POTEST.

Audore
L. EVLERO.

Conventui exhibita die 17 Iunii 1779.

§. I.

Principium, unde hae series sunt deductae, situm est in hac formula binomiali: $4 + x^4$, quam constat involvere hos duos factores rationales: $2 + 2x + xx$ et $2 - 2x + xx$. Hinc enim statim sequitur hanc formulam integram: $\int \frac{x(2 + 2x + xx)}{4 + x^4}$, quam signo \odot indicemus, reduci ad hanc: $\odot = \int \frac{2x}{2 - 2x + xx}$, cuius integrale, ita sumtum, ut evanescat posito $x = 0$, est $A \operatorname{tang.} \frac{x}{2 - x}$. Vbi observetur, casu $x = 1$ esse $\odot = \frac{\pi}{4}$; at vero casu $x = \frac{1}{2}$ erit $\odot = A \operatorname{tang.} \frac{1}{3}$; tum vero casu $x = \frac{1}{4}$ erit $\odot = A \operatorname{tang.} \frac{1}{7}$. Notum autem est esse

$$2 A \operatorname{tang.} \frac{1}{3} + A \operatorname{tang.} \frac{1}{7} = A \operatorname{tang.} 1 = \frac{\pi}{4}.$$

§. 2. Cum igitur formula integralis illa signo \odot indicata tribus constet partibus, singulas seorsim evolvamus, quas brev. gr. sequentibus characteribus insigniamus:

$$\text{I. } \int \frac{\partial x}{4+x^4} = \mathfrak{b}; \text{ II. } \int \frac{x \partial x}{4+x^4} = \mathfrak{z}; \text{ III. } \int \frac{x x \partial x}{4+x^4} = \mathfrak{o};$$

ita ut fit

$$\odot = 2 \mathfrak{b} + 2 \mathfrak{z} + \mathfrak{o} = A \text{ tang. } \frac{x}{2-x}.$$

Nunc igitur istas tres formulas integrales more solito in series infinitas evolvamus, inde formandas, quod fit

$$\frac{x}{4+x^4} = \frac{x}{4} \left(1 - \frac{x^4}{4} + \frac{x^8}{4^2} - \frac{x^{12}}{4^3} + \frac{x^{16}}{4^4} - \text{etc.} \right)$$

§. 3. Quod si iam primo istam seriem ducamus in ∂x et integremus, prima formula \mathfrak{b} persequentem seriem exprimetur:

$$\mathfrak{b} = \frac{x}{4} \left[1 - \frac{1}{5} \cdot \frac{x^4}{4} + \frac{1}{9} \left(\frac{x^4}{4} \right)^2 - \frac{1}{13} \left(\frac{x^4}{4} \right)^3 + \text{etc.} \right].$$

At vero illa series ducta in $x \partial x$ et integrata dabit

$$\mathfrak{z} = \frac{x x}{8} \left[1 - \frac{1}{3} \cdot \frac{x^4}{4} + \frac{1}{5} \left(\frac{x^4}{4} \right)^2 - \frac{1}{7} \left(\frac{x^4}{4} \right)^3 + \text{etc.} \right].$$

Dènique eadem series ducta in $x x \partial x$ et integrata praebet

$$\mathfrak{o} = \frac{x^3}{4} \left[\frac{1}{3} - \frac{1}{7} \cdot \frac{x^4}{4} + \frac{1}{11} \left(\frac{x^4}{4} \right)^2 - \frac{1}{15} \left(\frac{x^4}{4} \right)^3 + \text{etc.} \right].$$

§. 4. Cum igitur fit $\odot = 2 \mathfrak{b} + 2 \mathfrak{z} + \mathfrak{o}$, evolvamus seorsim casus initio memoratos, quibus est vel $x = 1$, vel $x = \frac{1}{2}$, vel $x = \frac{1}{4}$, quorum primo est $\frac{x^4}{4} = \frac{1}{4}$; secundo vero est $\frac{x^4}{4} = \frac{1}{64}$; tertio vero $\frac{x^4}{4} = \frac{1}{1024}$; unde patet binos casus posteriores maxime convergere; quin etiam ipsa prima, cuius termini in ratione quadrupla decrescunt, iam magis convergit quam series Leibnitiana, sumto arcu cuius tangens est

est $\frac{1}{\sqrt{3}}$, praeterquam quod hic calculus nulla irrationalitate perturbatur.

Evolutio casus primi.

quo $x = 1$ et $\odot = A \text{ tang. } 1 = \frac{\pi}{4}$.

§. 5. Cum igitur hic fit $\frac{x^4}{4} = \frac{1}{4}$, tres nostrae series principales \mathfrak{h} , \mathfrak{z} , \odot sequenti modo procedent:

$$\begin{aligned}\mathfrak{h} &= \frac{1}{4} \left[1 - \frac{1}{5} \cdot \frac{1}{4} + \frac{1}{9} \left(\frac{1}{4}\right)^2 - \frac{1}{13} \left(\frac{1}{4}\right)^3 + \frac{1}{17} \left(\frac{1}{4}\right)^4 - \text{etc.} \right] \\ \mathfrak{z} &= \frac{1}{8} \left[1 - \frac{1}{3} \cdot \frac{1}{4} + \frac{1}{5} \left(\frac{1}{4}\right)^2 - \frac{1}{7} \left(\frac{1}{4}\right)^3 + \frac{1}{9} \left(\frac{1}{4}\right)^4 - \text{etc.} \right] \\ \sigma &= \frac{1}{4} \left[\frac{1}{3} - \frac{1}{7} \cdot \frac{1}{4} + \frac{1}{11} \left(\frac{1}{4}\right)^2 - \frac{1}{15} \left(\frac{1}{4}\right)^3 + \frac{1}{19} \left(\frac{1}{4}\right)^4 - \text{etc.} \right]\end{aligned}$$

§. 6. Cum igitur fit $\odot = 2 \mathfrak{h} + 2 \mathfrak{z} + \sigma = \frac{\pi}{4}$, per 4 multiplicando valor ipsius π per sequentes tres series exprimitur

$$\pi = \begin{cases} 2 \left(1 - \frac{1}{5} \cdot \frac{1}{4} + \frac{1}{9} \cdot \frac{1}{4^2} - \frac{1}{13} \cdot \frac{1}{4^3} + \frac{1}{17} \cdot \frac{1}{4^4} - \text{etc.} \right) \\ 1 \left(1 - \frac{1}{3} \cdot \frac{1}{4} + \frac{1}{5} \cdot \frac{1}{4^2} - \frac{1}{7} \cdot \frac{1}{4^3} + \frac{1}{9} \cdot \frac{1}{4^4} - \text{etc.} \right) \\ 1 \left(\frac{1}{3} - \frac{1}{7} \cdot \frac{1}{4} + \frac{1}{11} \cdot \frac{1}{4^2} - \frac{1}{15} \cdot \frac{1}{4^3} + \frac{1}{19} \cdot \frac{1}{4^4} - \text{etc.} \right) \end{cases}$$

§. 7. Ex his certe ternis seriebus ratio peripheriae ad diametrum multo minore labore computari potuisset quam ex serie *Leibnitiana*, qua Auctores illi meritissimi *Sharp*, *Machin* & *Lagny* sunt usi, quorum primus valorem ipsius π in fractione decimali usque ad 72 figuras, secundus ad 100, ac postremus adeo usque ad 128 determinavit. At vero sequentes casus multo magis istum laborem sublevare possent.

Evo-

Evolutio casus secundi,

quo $x = \frac{1}{2}$.

§. 8. Hoc igitur casu erit $\frac{x^4}{4} = \frac{1}{64}$, unde tres illae series sequenti modo referentur:

$$\begin{aligned} \mathfrak{h} &= \frac{1}{8} \left(1 - \frac{1}{5} \cdot \frac{1}{64} + \frac{1}{9} \cdot \frac{1}{64^2} - \frac{1}{13} \cdot \frac{1}{64^3} + \text{etc.} \right) \\ \mathfrak{z} &= \frac{1}{32} \left(1 - \frac{1}{3} \cdot \frac{1}{64} + \frac{1}{5} \cdot \frac{1}{64^2} - \frac{1}{7} \cdot \frac{1}{64^3} + \text{etc.} \right) \\ \sigma &= \frac{1}{32} \left(\frac{1}{3} - \frac{1}{7} \cdot \frac{1}{64} + \frac{1}{11} \cdot \frac{1}{64^2} - \frac{1}{15} \cdot \frac{1}{64^3} + \text{etc.} \right) \end{aligned}$$

§. 9. Cum igitur $2\mathfrak{h} + 2\mathfrak{z} + \sigma = A \text{ tang. } \frac{1}{3}$, erit

$$A \text{ tang. } \frac{1}{3} = \left\{ \begin{aligned} &\frac{1}{4} \left(1 - \frac{1}{5} \cdot \frac{1}{64} + \frac{1}{9} \cdot \frac{1}{64^2} - \frac{1}{13} \cdot \frac{1}{64^3} + \text{etc.} \right) \\ &\frac{1}{16} \left(1 - \frac{1}{3} \cdot \frac{1}{64} + \frac{1}{5} \cdot \frac{1}{64^2} - \frac{1}{7} \cdot \frac{1}{64^3} + \text{etc.} \right) \\ &\frac{1}{32} \left(\frac{1}{3} - \frac{1}{7} \cdot \frac{1}{64} + \frac{1}{11} \cdot \frac{1}{64^2} - \frac{1}{15} \cdot \frac{1}{64^3} + \text{etc.} \right) \end{aligned} \right\}$$

Eti autem hic tres computandae sunt series, tamen, quia singulae secundum eandem rationem $1:64$ decrescunt, laborem mirum in modum contrahere licebit.

Evolutio casus tertii,

quo $x = \frac{1}{4}$.

§. 10. Cum igitur hic fit $\frac{x^4}{4} = \frac{1}{1024}$, series nostrae tres principales ita se habebunt:

$$\begin{aligned} \mathfrak{h} &= \frac{1}{16} \left(1 - \frac{1}{5} \cdot \frac{1}{1024} + \frac{1}{9} \cdot \frac{1}{1024^2} - \frac{1}{13} \cdot \frac{1}{1024^3} + \text{etc.} \right) \\ \mathfrak{z} &= \frac{1}{128} \left(1 - \frac{1}{3} \cdot \frac{1}{1024} + \frac{1}{5} \cdot \frac{1}{1024^2} - \frac{1}{7} \cdot \frac{1}{1024^3} + \text{etc.} \right) \\ \sigma &= \frac{1}{256} \left(\frac{1}{3} - \frac{1}{7} \cdot \frac{1}{1024} + \frac{1}{11} \cdot \frac{1}{1024^2} - \frac{1}{15} \cdot \frac{1}{1024^3} + \text{etc.} \right) \end{aligned}$$

§. 11. Cum igitur $2\frac{1}{3} + 2\frac{1}{4} + \sigma = A \text{ tang. } \frac{1}{7}$, erit
his seriebus debite iunctis:

$$A \text{ tang. } \frac{1}{7} = \left\{ \begin{array}{l} \frac{1}{8} \left(1 - \frac{1}{5} \cdot \frac{1}{1024} + \frac{1}{9} \cdot \frac{1}{1024^2} - \frac{1}{13} \cdot \frac{1}{1024^3} + \text{etc.} \right) \\ \frac{1}{64} \left(1 - \frac{1}{3} \cdot \frac{1}{1024} + \frac{1}{5} \cdot \frac{1}{1024^2} - \frac{1}{7} \cdot \frac{1}{1024^3} + \text{etc.} \right) \\ \frac{1}{256} \left(\frac{1}{3} - \frac{1}{7} \cdot \frac{1}{1024} + \frac{1}{11} \cdot \frac{1}{1024^2} - \frac{1}{15} \cdot \frac{1}{1024^3} + \text{etc.} \right) \end{array} \right\}$$

Applicatio binorum casuum posteriorum ad peripheri-
am circuli per series maxime convergentes ex-
primendam.

§. 12. Cum fit, uti iam observavimus,

$\frac{\pi}{4} = 2A \text{ tang. } \frac{1}{3} + A \text{ tang. } \frac{1}{7}$ erit $\pi = 8A \text{ tang. } \frac{1}{3} + 4A \text{ tang. } \frac{1}{7}$
seriebus supra inventis substitutis valor ipsius π per sex se-
quentes series coniunctim exprimetur:

$$\pi = \left\{ \begin{array}{l} 2 \left(1 - \frac{1}{5} \cdot \frac{1}{64} + \frac{1}{9} \cdot \frac{1}{64^2} - \frac{1}{13} \cdot \frac{1}{64^3} + \text{etc.} \right) \\ \frac{1}{2} \left(1 - \frac{1}{3} \cdot \frac{1}{64} + \frac{1}{5} \cdot \frac{1}{64^2} - \frac{1}{7} \cdot \frac{1}{64^3} + \text{etc.} \right) \\ \frac{1}{4} \left(\frac{1}{3} - \frac{1}{7} \cdot \frac{1}{64} + \frac{1}{11} \cdot \frac{1}{64^2} - \frac{1}{15} \cdot \frac{1}{64^3} + \text{etc.} \right) \\ \frac{1}{2} \left(1 - \frac{1}{5} \cdot \frac{1}{1024} + \frac{1}{9} \cdot \frac{1}{1024^2} - \frac{1}{13} \cdot \frac{1}{1024^3} + \text{etc.} \right) \\ \frac{1}{16} \left(1 - \frac{1}{3} \cdot \frac{1}{1024} + \frac{1}{5} \cdot \frac{1}{1024^2} - \frac{1}{7} \cdot \frac{1}{1024^3} + \text{etc.} \right) \\ \frac{1}{64} \left(\frac{1}{3} - \frac{1}{7} \cdot \frac{1}{1024} + \frac{1}{11} \cdot \frac{1}{1024^2} - \frac{1}{15} \cdot \frac{1}{1024^3} + \text{etc.} \right) \end{array} \right\}$$

Hic ergo maxime notatu dignum occurrit, quod omnes is-
tae series per solas potestates binarii procedant.